

Time Reversal Symmetry Breaking Holographic Superconductor in Constant External Magnetic Field

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Abstract

It is known that a classical $SU(2)$ Einstein-Yang-Mills theory in 3+1 dimensional anti-de Sitter spacetime can provide a holographic dual to a 2+1 dimensional time reversal symmetry breaking superconductor with a pseudogap. We study the properties of this holographic superconductor in the presence of an applied constant external magnetic field, neglecting backreaction on the geometry. The superconductor is immersed into a constant external magnetic field by adding a radially (the extra dimension) dependent magnetic field to the black hole. As for real superconductors, there is a critical magnetic field above which no superconductivity can appear. The continuity of the first derivative of the free energy difference between the superconducting phase and the normal phase at the critical temperature suggests that the superconducting phase transition with applied magnetic field is of second order.

I. INTRODUCTION

As a powerful tool to understand strong coupled gauge theories, AdS/CFT correspondence [1, 2, 3, 4] has been applied in condensed matter physics recently. There are many attempts to use this gauge/gravity correspondence to describe certain condensed matter phenomenons such as the (classical) Hall effect [5], the Nernst effect [6, 7, 8], the quantum Hall effect [9, 10, 11], superconductivity [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25] and superfluid [26].

Superconductivity is a common phenomenon occurring in certain materials, characterized by exact zero electrical resistance and the exclusion of the interior magnetic field (the Meissner effect). The essence of superconductivity is the spontaneous breaking of a local $U(1)$ gauge symmetry due to a charged condensate, which is formed by Cooper pairs of electrons in the BCS theory. Models of black holes coupled to matter fields also exhibit spontaneous symmetry breaking solutions. Gubser [12] has presented an argument that by coupling the Abelian Higgs model to gravity with a negative cosmological constant, one gets solutions which spontaneously break the Abelian gauge symmetry via a charged complex scalar condensate near the horizon of the black hole. An Einstein-Yang-Mills (EYM) model with fewer parameters whose Lagrangian is mostly determined by symmetry principles is constructed later by Gubser [14] and is shown to have spontaneous symmetry breaking solutions due to a condensate of non-Abelian gauge fields in the theory. Hartnoll *et al* [13] explored further the connections of superconductors and black holes and built a holographic (in the sense of AdS/CFT duality) superconductor which exhibits the basic features of a superconductor such as the existence of a critical temperature below which a charged condensate forms.

Features that are not those of BCS theory are also captured by this holographic description of superconductivity. Gubser and Pufu [17] studied a model of superconducting black holes with p -wave gap solutions whose order parameter is a vector. Roberts and Hartnoll [18] found two major nonconventional features for the holographic superconductor whose dual theory involves a classical $SU(2)$ EYM theory, one of which is a pseudogap at zero temperature, and the other is the spontaneous breaking of time reversal invariance.

While the spontaneous breaking of time reversal invariance is exhibited by a nonvanishing Hall conductivity in the absence of an external magnetic field, our purpose in this paper is to study the behavior of this time reversal symmetry breaking holographic superconductor

in the presence of a constant external magnetic field. This external magnetic field is implemented on the superconductor by considering a magnetically charged Reissner-Nordström black hole rather than a Schwarzschild black hole in anti-de Sitter (AdS) spacetime in the gravity sector. Our numerical investigations show that the critical temperature decreases with the increasing of the magnetic field and there is a critical magnetic field above which no phase transition occurs. The phase transition at the critical temperature is a second order one, as suggested by the continuity of the first derivative of the free energy difference between the superconducting phase and the normal phase.

The organization of this paper is as follows. In Sec. II, we first review the EYM theory which is dual to the time reversal symmetry breaking superconductor and then discuss the effect of adding an external magnetic field to it. We obtain in this section analytically the negative effective mass of the condensate, which is a sign of spontaneous symmetry breaking solutions to the model. Section III is devoted to the numerical studies of this model. Section IV concludes and gives some further discussions.

II. MODEL OF HOLOGRAPHIC SUPERCONDUCTOR IN EXTERNAL MAGNETIC FIELD

The effective physics of many unconventional superconductors such as cuprates is in 2+1 dimensional spacetime. We will focus in this paper on a 2+1 dimensional superconductor system which is a gauge field theory whose holographic dual is a 3+1 dimensional theory with gravity in AdS spacetime (AdS₄/CFT₃).

The starting point of studying the holographic superconductor at finite temperature T is choosing a black hole solution with a negative cosmological constant so that the Hawking temperature of the black hole is T . The full action of the EYM theory in 3+1 dimensional spacetime considered in [18] consists of two sectors, the gravity sector and the matter sector,

$$S_{\text{EYM}} = \int \sqrt{-g} d^4x \left[\frac{1}{2\kappa_4^2} \left(R + \frac{6}{L^2} \right) - \frac{1}{2g_{\text{YM}}^2} \text{Tr}(F_{\mu\nu} F^{\mu\nu}) \right], \quad (1)$$

where g_{YM} is the gauge coupling constant and $F_{\mu\nu} = T^a F_{\mu\nu}^a = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$ is the field strength of the gauge field $A = A_\mu dx^\mu = T^a A_\mu^a dx^\mu$. For the $SU(2)$ gauge symmetry, $[T^a, T^b] = i\epsilon^{abc} T^c$ and $\text{Tr}(T^a T^b) = \delta^{ab}/2$, where ϵ^{abc} is the totally antisymmetric tensor with $\epsilon^{123} = 1$. The Yang-Mills Lagrangian becomes $\text{Tr}(F_{\mu\nu} F^{\mu\nu}) = F_{\mu\nu}^a F^{a\mu\nu}/2$ with the field

strength components $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \epsilon^{abc} A_\mu^b A_\nu^c$.

Working in the probe limit in which the matter fields do not back react on the metric as in [17, 18] and taking the planar Schwarzschild-AdS ansatz, the black hole metric reads (We use mostly plus signature for the metric.)

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(dx_1^2 + dx_2^2), \quad (2)$$

where the metric function $f(r)$ is

$$f(r) = \frac{r^2}{L^2} - \frac{M}{r}. \quad (3)$$

L and M are the radius of the AdS spacetime and the mass of the black hole, respectively. They determine the Hawking temperature of the black hole,

$$T = \frac{3M^{1/3}}{4\pi L^{4/3}}, \quad (4)$$

which is also the temperature of the dual gauge theory living on the boundary of the AdS spacetime.

The equations of motion for the gauge fields can be obtained by using the Euler-Lagrange equations to be

$$\frac{1}{\sqrt{-g}}\partial_\mu (\sqrt{-g}F^{a\mu\nu}) + \epsilon^{abc}A_u^b F^{c\mu\nu} = 0. \quad (5)$$

Considering the ansatz [14, 18]

$$A = \phi(r)T^3dt + w(r)(T^1dx^1 + T^2dx^2) \quad (6)$$

for the gauge fields, the Yang-Mills Lagrangian density reads [Note that $\sqrt{-g} = r^2$, as can be seen from (2).]

$$\mathcal{L}_{\text{YM}} = -\frac{\sqrt{-g}}{2g_{\text{YM}}^2}\text{Tr}(F_{\mu\nu}F^{\mu\nu}) = -\frac{1}{4g_{\text{YM}}^2}\left(-\frac{2}{f}\phi^2w^2 + 2fw'^2 - r^2\phi'^2 + \frac{w^4}{r^2}\right). \quad (7)$$

From this Lagrangian density, we can derive the following equations of motion,

$$\phi'' + \frac{2}{r}\phi' - \frac{2w^2}{r^2f}\phi = 0, \quad (8)$$

and

$$w'' + \frac{f'}{f}w' + \frac{\phi^2}{f^2}w - \frac{1}{r^2f}w^3 = 0, \quad (9)$$

which can also be deduced from (5) and (6).

The $U(1)$ subgroup of $SU(2)$ generated by T^3 is identified to be the electromagnetic gauge group [14] and ϕ is the electrostatic potential, which must vanish at the horizon for the gauge field A to be well defined, but need not vanish at infinity. Thus the black hole can carry charge through the condensate w , which spontaneously breaks the $U(1)$ gauge symmetry. From the Yang-Mills Lagrangian density (7), we can see that the effective mass (along the radial direction r) for w is

$$m_{eff}^2 = -\phi^2/f < 0, \quad (10)$$

which is characteristic to spontaneous symmetry breaking theories.

In this paper, we are interested in the properties of this kind of holographic superconductor with an applied external magnetic field. To achieve this, we consider a magnetically charged Reissner-Nordström black hole rather than a Schwarzschild-AdS black hole in anti-de Sitter space-time (RNAdS) in the gravity sector [15, 16, 27]. Now the full EYM action becomes

$$S'_{\text{EYM}} = \int \sqrt{-g} d^4x \left[\frac{1}{2\kappa_4^2} \left(R + \frac{6}{L^2} \right) - \frac{1}{4} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} - \frac{1}{2g_{\text{YM}}^2} \text{Tr}(F_{\mu\nu} F^{\mu\nu}) \right], \quad (11)$$

where the first two terms belong to the gravity sector. By setting the nonvanishing components of the field strength of the Maxwell field to be $\mathcal{F}_{xy} = r^2 H = -\mathcal{F}_{yx}$ such that $-\mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}/4 = -H^2/2$, we immerse the superconductor at the AdS boundary into a perpendicular constant external magnetic field.

We still work in the probe limit [17, 18] where the metric is only determined by the gravity sector. In this limit, the background geometry is given by the magnetically charged RNAdS black hole with a metric taking the same form as in (2), but with the metric function (3) being replaced by

$$f(r) = \frac{r^2}{L^2} - \frac{M}{r} + \frac{H^2}{r^2}. \quad (12)$$

The Hawking temperature of the RNAdS black hole is determined by

$$T = \frac{f'(r_+)}{4\pi}, \quad (13)$$

where r_+ is the radius of the outer horizon of the RNAdS black hole. We notice that for the vanishing magnetic field $H = 0$, the metric function $f(r)$ in (12) and the Hawking temperature T in (13) reduce to the previous ones in (3) and (4), respectively.

In the probe limit, by adding an external magnetic field, the black hole background considered is changed, and the equations of motion for the non-Abelian gauge fields have the same forms as in (8) and (9) with the metric function $f(r)$ replaced by (12). Our task in the next section is to find solutions to the equations of motion (8) and (9) of the fields ϕ and w with appropriate boundary conditions to see how the applied magnetic field affects the dual superconductor.

III. SOLUTIONS TO THE MODEL

There are two solutions to the equations of motion of the fields ϕ and w , one without hair and the other with hair, corresponding to the normal and the superconductor states of the system, respectively.

The no hair solution is easy to find by setting $w = 0$, which gives $\phi = \mu - \rho/r$, where μ and ρ are the chemical potential and the charge density of the field theory, respectively.

The hairy solution can only be found by numerically solving the second order nonlinear equations (8) and (9) with the boundary condition $\phi(r_+) = 0$ for $\phi(r)$ at the outer horizon and the following asymptotic behaviors of the fields at the AdS conformal boundary $r \rightarrow \infty$ [18],

$$\phi = \mu - \frac{\rho}{r} + \cdots, \quad (14)$$

$$w = w_0 + \frac{w_1}{r} + \cdots = \frac{\langle J \rangle}{\sqrt{2}r} + \cdots, \quad (15)$$

where $\langle J \rangle$ is the condensate of the charged operator dual to the field w and is the order parameter for the superconductivity phase. The constant term w_0 vanishes, since there is no source term in the field theory action for the operator $\langle J \rangle$ [17, 18].

Properties of the dual field theory can be read off from the above asymptotic behaviors via the AdS/CFT correspondence. The strategy to find the above hairy black hole solution is to expand the fields near the outer horizon and numerically integrate them out from the horizon to infinity. To facilitate the numerical calculations, we set the AdS radius to be $L = 1$, which can be achieved by using the rescaling symmetry of the model. In the absence of a magnetic field, there is only one horizon r_0 , which can be set to be 1 by the conformal properties, and one can develop power series solutions for ϕ and w near the horizon $z = 1$ where z is the inverse of r . This change of variable from r to z brings much convenience to numerical

studies. However, with the introduction of the magnetic field, the outer horizon sits at $z = \frac{1}{r_+} > 1$, making the series expansions of the fields near the outer horizon much more complicated and disabling Mathematica for doing the subsequent numerical calculations. We circumvent this difficulty by using another change of variable from z to $y = r_+ z$ such that power series solutions for ϕ and w can be developed near the horizon $y = 1$.

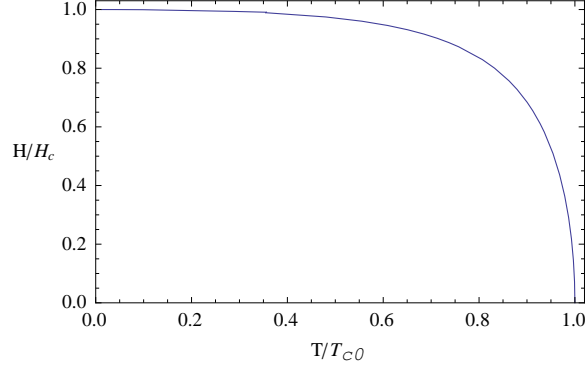


FIG. 1: Dependence of the critical temperature T_c (along the line in the diagram) on the applied magnetic field H . T_{c0} and H_c are the critical temperature without the applied magnetic field ($H = 0$) and the critical magnetic field at zero temperature, respectively.

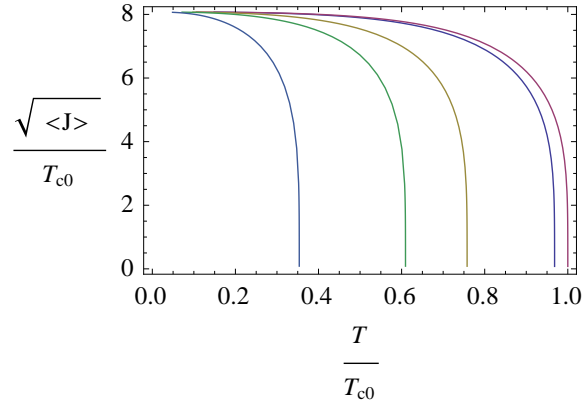


FIG. 2: The order parameters $\sqrt{\langle J \rangle}$ as functions of the temperature for different magnitudes of the applied magnetic field H . T_{c0} is the critical temperature without the applied magnetic field. The lines from right to left correspond to the cases $H = 0, 0.3, 0.6, 0.65$, and 0.68 , respectively.

We have studied systematically the numerical solutions with different magnitudes of the magnetic field H . We found that there is a hair solution for any H in the range $0 \leq H < H_{\max}$, where $H_{\max} = 0.687365$ is the solution to the equation $27 - 256H^6 = 0$ and

is the maximal value the magnetic field can take in order for the black hole to be censored by at least one horizon. It turns out that the critical temperature T_c decreases with the increasing of the magnetic field H . See Figs. 1 and 2 for illustrations. The decreasing of the T_c is very slow when H is small and gets more and more rapid when H becomes larger and larger. Finally, T_c vanishes when H reaches the critical value $H_c = H_{\max}$.

Important information about the phase transition can be extracted from the behavior of the free energy, as in the studies on holographic superfluidity by Herzog *et al* [26]. The free energy of the field theory is determined by the value of the Yang-Mills action (ignoring the backreaction of the gauge fields on the metric)

$$S_{\text{YM}} = \int d^4x \mathcal{L}_{\text{YM}} \quad (16)$$

evaluated on shell up to boundary counterterms, $F = -TS_{\text{os}} + \dots$, where the ellipsis denotes boundary terms that we should introduce to regulate the action if needed. The on-shell Yang-Mills action S_{os} is determined by plugging the equations of motion (8) and (9) into the explicit form of the Yang-Mills Lagrangian (7) (omitting the irrelevant factor $1/4g_{\text{YM}}^2$),

$$S_{\text{os}} = \int d^3x (-\phi\phi' + 2z^2 f w w')|_{z=\epsilon} - \int d^3x \int_{\epsilon}^{z_h} dz \left(-w^4 + \frac{2\phi^2 w^2}{f z^2} \right), \quad (17)$$

where the metric function f in (17) should be understood as a function of the inverse radial (or Fefferman-Graham) coordinate $z = 1/r$, $f(z) = 1/L^2 z^2 - zM + H^2 z^2$ and $\epsilon = 0^+$ and z_h are the boundary of the AdS spacetime and the outer horizon of the RNAdS black hole, respectively.

To regulate S_{os} , it is important to choose an ensemble. By keeping μ fixed, we are working in the grand canonical ensemble without an additional boundary term [26]. Near the boundary $z = \epsilon$, the fields ϕ and w are determined by (14) and (15) and the two terms $-\phi\phi'$ and $2z^2 f w w'$ in (17) give $\mu\rho$ and $2w_0 w_1$, respectively. We can see that the on-shell action S_{os} is not divergent and no counterterms are needed. Since w_0 is fixed to be zero, for a spatially homogenous system, the free energy density of the field theory takes the following form,

$$F/V = -\mu\rho + \int_{\epsilon}^{z_h} dz \left(-w^4 + \frac{2\phi^2 w^2}{f z^2} \right), \quad (18)$$

where $V \equiv \int d^3x$.

Figure 3 shows the difference of the free energy density $\Delta F/V$ between the superconducting phase and the normal phase as a function of temperature T for two different values of the

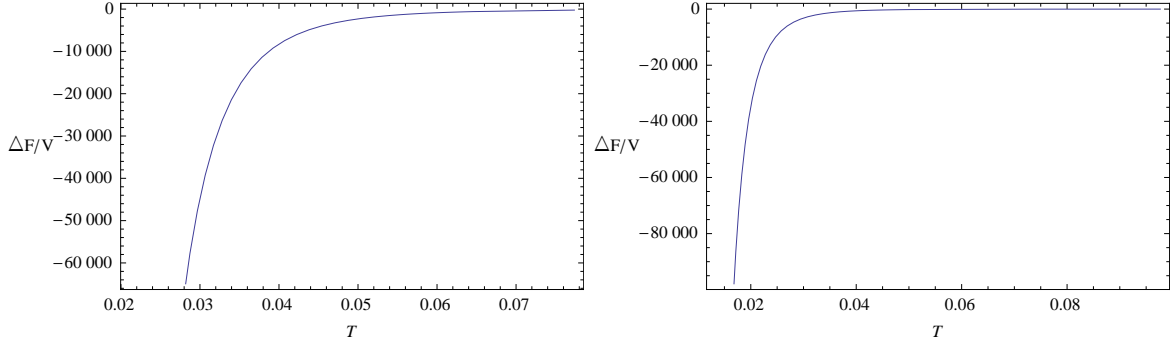


FIG. 3: The difference in free energy density $\Delta F/V$ between the superconducting phase and the normal phase as a function of the temperature T for $H = 0.2$ (left diagram) and $H = 0.6$ (right diagram).

external magnetic field H . As can be seen, the free energy density differences are smooth at the critical temperatures (the first derivatives of the free energy differences are continuous at T_c) which suggests that the phase transitions at the critical temperatures are of second order. When the magnetic field vanishes, $H = 0$, the critical temperature $T_c = T_{c0}$, and we get similar results which are consistent with those given in [18].

IV. CONCLUSION AND DISCUSSION

In this paper, we have studied the effects of an external magnetic field for a holographic superconductor dual to an EYM theory. A critical magnetic field exists for this model. Below this critical magnetic field, a condensate develops and triggers superconductivity. The critical magnetic field corresponds to the maximal magnetic charge the black hole can carry such that the black hole is censored by a horizon. This is consistent with the concept of the hairy black hole that the spontaneous symmetry breaking of the gauge invariance which results in superconductivity occurs slightly outside the horizon [12].

The existence of a critical magnetic field at a given critical temperature for the holographic superconductor considered in this work is expected for real-world superconductors. The critical temperature drops as the external magnetic field increases. Similar conclusion is obtained by considering a rotating holographic superconductor with a dual gravity background given by the Kerr-Newman-AdS solution [25]. The critical temperature is found to decrease as the angular momentum of the dual black hole increases. It would be interesting

to consider the effects of adding both magnetic field and angular momentum to the dual black hole.

The existence of a superconductivity phase in spite of the presence of an external magnetic field is a signal of the Meissner effect. However, our simple model can not distinguish between type I and type II superconductors. The ansatz (6) considered in this paper has a symmetric form for the $SU(2)$ gauge fields $A_1^1 = A_2^2 = w$. Interesting results may be obtained by considering a less symmetric ansatz such as $A_1^1 = w, A_2^2 = 0$, the one studied in [17].

V. ACKNOWLEDGEMENT

We thank professors S. A. Hartnoll and C. P. Herzog for great help. We also thank G. Chen, P. Kerner and J. Sonner for helpful comments. This work is supported by the National Natural Science Foundation of China under Grants No. 10535010, 10675090, 10775068, and 10735010.

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